Lecture 4 - Electroweak Theory

This lecture will explain the $SU(2)\times U(1)$ sector of the SM, including the SM Higgs mechanism and the couplings to fermions.

We begin by studying the structure of the $SU(2)$ non-Abelian gauge theory.

SU(2)

$G = SU(2)$

we study the fundamental rep since we want to couple to matter

$[\tau^a, \tau^b] = i\varepsilon^{abc}\tau^c$, $a = 1 \ldots 3$, $\varepsilon^{123} = 1$, $\varepsilon^{132} = -1$ ...

The generators in the fundamental rep are

$$\tau^a = \frac{\sigma^a}{2} \text{ w/ } \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

3 generators $\Rightarrow$ 3 gauge fields $W_\mu = W_\mu^a \tau^a$

(use name "W" since these will be related to SM W-bosons)

Under $SU(2)$ gauge transformation $\Omega = e^{i\alpha_a(x)\tau_a}$,

we have $W_\mu \to \Omega W_\mu \Omega^{-1} + i\Omega\partial_\mu\Omega^{-1}$

Let's explore the properties of W_μ

We can write the matrix explicitly as

$$W_\mu = \frac{1}{2}\begin{pmatrix} W_\mu^3 & W_\mu^1 - iW_\mu^2 \\ W_\mu^1 + iW_2^\mu & -W_\mu^3 \end{pmatrix}$$

Let us focus on the subset of transformation parameters with $\alpha_1 = \alpha_2 = 0$

This isolates the $U(1)$ subgroup of $SU(2)$

This is closely related to $U(1)_{EM}$ as we will eventually see.

Addionally take $\alpha_3 = \text{const}$ and infinitesimal

$$\Rightarrow W_\mu \rightarrow e^{i\alpha\tau_3} W_\mu e^{-i\alpha\tau_3} \simeq W_\mu + i\alpha[\tau_3, W_\mu]$$

$$= W_\mu + i\alpha \frac{1}{2}\begin{pmatrix} 0 & W_1 - iW_2 \\ -(W_1 + iW_2) & 0 \end{pmatrix}$$

So some of the W_μ^a's transform under this $U(1)$ subgroup $\Rightarrow$ in SM some of them will have electric charge.

This is inevietable since $U(1) \subset SU(2)$ so $U(1)$ generator τ_3 does <u>not</u> commute w/ all $SU(2)$ generators

Since we expect a field ϕ to transform under a $U(1)$ symmetry as $\delta\varphi = iQ\alpha\varphi$ for charge Q and infinitesimal α

we can identify:

$$W^3_\mu : \delta W^3_\mu = 0 \Rightarrow \text{"neutral"}$$

$$W^+_\mu \equiv \frac{W^1_\mu - iW^2_\mu}{\sqrt{2}} : \delta W^+_\mu = +\alpha W^+_\mu \Rightarrow \text{"charge } +1\text{"}$$

$$W^-_\mu \equiv \frac{W^1_\mu + iW^2_\mu}{\sqrt{2}} : \delta W^-_\mu = -\alpha W^-_\mu \Rightarrow \text{"charge } -1\text{"}$$

So we have a charged vector field.

Note the $1/\sqrt{2}$ is a normalization convention to ensure a canonical kinetic term.

As we will see, the neutral vector will be part of the story for the photon and Z^0 boson (which are both electrically neutral)

Let us write down the Lagrangian in terms of $W^{+/-}$ and W^3 (you will derive this in H/W)

$$\mathcal{L} = \mathcal{L}_{kin} + \mathcal{L}_3 + \mathcal{L}_4$$

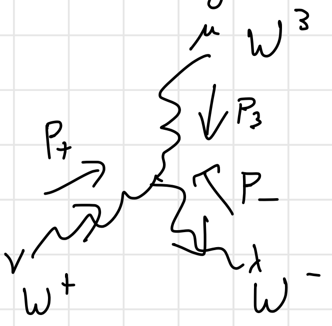
w/ $$\mathcal{L}_{kin} = -\frac{1}{4}\left(W^3_{\mu\nu}\right)^2 - \frac{1}{2}\left|W^+_{\mu\nu}\right|^2$$

where $V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$

$$\mathcal{L}_3 = +ig\left[(\partial_\mu W^-_\nu)W^{+\nu}W^{3\mu} - (\partial_\mu W^+_\nu)W^{-\nu}W^{3\mu}\right]$$
$$+ig\left[(\partial_\mu W^3_\nu)W^{-\nu}W^{+\mu} - (\partial_\mu W^-_\nu)W^{3\nu}W^{+\mu}\right]$$
$$+ig\left[(\partial_\mu W^+_\nu)W^{3\nu}W^{-\mu} - (\partial_\mu W^3_\nu)W^{+\nu}W^{-\mu}\right]$$

$$\mathcal{L}_4 = g^2\left(W^3_\mu W^{+\mu}\right)\left(W^3_\nu W^{-\nu}\right) - g^2\left(W^3_\mu W^{3\mu}\right)\left(W^+_\mu W^{-\mu}\right)$$
$$+\frac{g^2}{2}\left(W^+_\mu W^{+\mu}\right)\left(W^-_\nu W^{-\nu}\right) - \frac{g^2}{2}\left(W^+_\mu W^{-\mu}\right)\left(W^+_\nu W^{-\nu}\right)$$

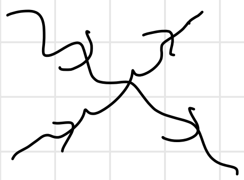
This gives 3-point Feynman vertices like

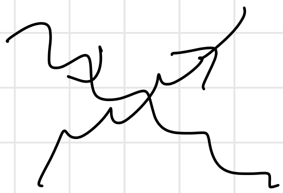


$$= -ig\left[(p_+ - p_-)_\mu \eta_{\nu\lambda} + (p_- - p_3)_\nu \eta_{\mu\lambda} + (p_3 - p_+)_\lambda \eta_{\mu\nu}\right]$$

Note arrows: flipping arrow sends $W^{+/-} \rightarrow W^{-/+}$

and 4-point vertices like

 $\Rightarrow$ 4 charged W interactions

 $\Rightarrow$ 2 charged W, 2 neutral W interactions

Note there are "missing" verticies that are allowed by simply considering "charge conservation". In particular, we do not have any $(W^3)^3$ or $(W^3)^4$ interactions. This is because of the asymmetry of the structure constants (there are no self-interactions of a single gauge field)

SU(2) coupled to matter

Let us introduce a Dirac fermion that transforms under the fundamental rep of SU(2), a "doublet"

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \quad \text{w/} \quad \psi \xrightarrow{SU(2)} e^{i\alpha^a \tau^a} \psi$$

The free Lagrangian is

$$\mathcal{L}^0 = \bar{\psi}_1 i\not{\partial} \psi_1 - m \bar{\psi}_1 \psi_1 + \bar{\psi}_2 i\not{\partial} \psi_2 - m \bar{\psi}_2 \psi_2$$

$$= \bar{\psi} i\not{\partial} \psi - m \bar{\psi} \psi$$

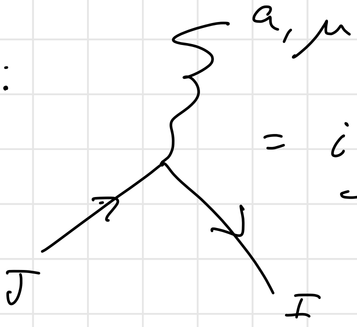
Note that $m_1 = m_2 = m$ to be consistent w/ $SU(2)$

The fermion interactions with $SU(2)$ gauge bosons are fully specified by prescription $\partial_\mu \to D_\mu$

w/ $D_\mu \psi = \partial_\mu \psi - i g W_\mu^a \tau^a \psi$

$$\Rightarrow \mathcal{L}_{int} = g \bar{\psi} \gamma^\mu \tau^a \psi W_\mu^a = g (\tau^a)_I{}^J \bar{\psi}^I \gamma^\mu \psi_J W_\mu^a$$

So we have a vertex:



$$= i g (\tau^a)_I{}^J \gamma^\mu$$

Think of the τ matrix as the "non-Abelian charge"

Note that transformation rules for non-Abelian matter are very restrictive.

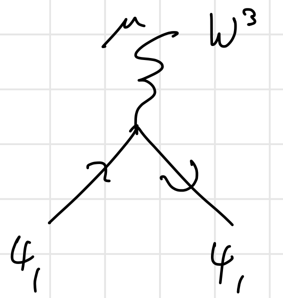
Let us write the interactions more explicitly in terms of $W_\mu^{+/-}$ and W_μ^3. Note that since $\tau_3 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$, the two components of the doublet have charge

$$Q(\psi_1) = +1/2 \quad \text{and} \quad Q(\psi_2) = -1/2$$

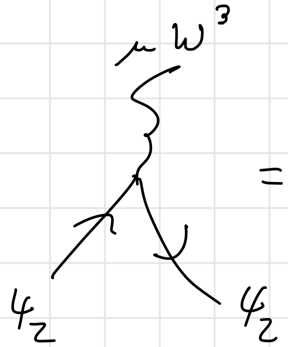
Then using $W_\mu = \frac{1}{2}\begin{pmatrix} W_\mu^3 & \sqrt{2}\, W_\mu^+ \\ \sqrt{2}\, W_\mu^- & -W_\mu^3 \end{pmatrix}$

$$\Rightarrow \mathcal{L}_{int} = \frac{g}{2}\bar{\psi}_1 \not{W}^3 \psi_1 - \frac{g}{2}\bar{\psi}_2 \not{W}^3 \psi_2$$
$$+ \frac{g}{\sqrt{2}}\bar{\psi}_1 \not{W}^+ \psi_2 + g\,\bar{\psi}_2 \not{W}^- \psi_1$$

So we have interactions w/ W^3 proportional to the fermion's charge:

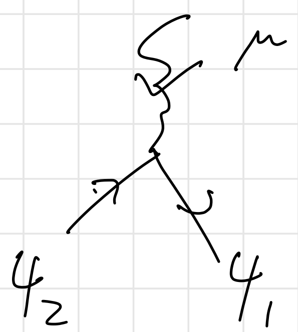


$$= i\frac{g}{2}\gamma^\mu$$

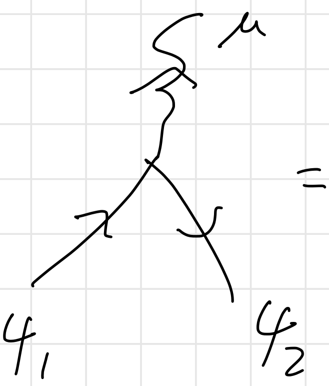


$$= -i\frac{g}{2}\gamma^\mu$$

and



$= i\frac{g}{\sqrt{2}}\gamma^\mu$



$= i\frac{g}{\sqrt{2}}\gamma^\mu$

These are "charged current interactions" since they induce interactions between different components of the doublet (that have different charges).

Higgs Mechanism in the Standard Model

Want a model that gives 3 massive vectors and one massless one. Experimental input tells us that weak interactions give "charged current" and "neutral current" transitions, and we need a photon.

One option (which is the right one) is to spontaneously break $SU(2) \times U(1)$ with a scalar field that transforms in the fundamental representation.

We will refer to the $SU(2)$ as $SU(2)_L$ where "L" is for "left" (this will become clear when we couple to fermions).

coupling g

We will refer to the $U(1)$ as $U(1)_Y$ where "Y" is called "hypercharge".

coupling g'

(This name is historical, going back to early work classifying hadrons.)

Note $U(1)_Y \neq U(1)_{EM}$. In fact, the SM Higgs mechanism results in the breaking pattern $SU(2)_L \times U(1)_Y \to U(1)_{EM}$.

To see how this works, we introduce the "Higgs Doublet" $H = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}$

It transforms in the fundamental rep of $SU(2)$. We assign it hypercharge $Y(H) = \frac{1}{2}$

(Note there are different conventions for hypercharge.)

Note that at this point, the hypercharge of the Higgs doublet is arbitrary (as long as it is non-zero). Only relative $U(1)$ charge is physical since we can rescale g'.

Assume the potential for the Higgs has the wine bottle shape: $V = \lambda\left(|H|^2 - \frac{\mu^2}{2\lambda}\right)^2$

$$\Rightarrow \sqrt{|\langle H_1\rangle|^2 + |\langle H_2\rangle|^2} = \frac{\mu}{\sqrt{2\lambda}} \equiv \frac{v}{\sqrt{2}} \neq 0$$

Choose to expand the theory around

$$\langle H\rangle = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}$$

Recall from discussion of pure $SU(2)$ theory that the $U(1)$ subgroup is $\tau^3 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$

Note τ^3 acts non-trivially on this vev.

However, the combination $Q = T^3 + Y$ where T^3 is eigenvalue of τ^3 and Y is hypercharge evaluates on H to

$$Q = \begin{pmatrix} \frac{1}{2}+\frac{1}{2} & 0 \\ 0 & \frac{1}{2}-\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

So the charge of the vacuum $\langle H\rangle = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}$ is zero. This is the unbroken $U(1)$ which we identify with electromagnitism. $U(1)_Q$

Note, we can still use our calculations from | 4.11

The $SU(2)$ theory because G_{SM} is direct product of $SU(2)_L$ and $U(1)_Y$, so gauge transformations take the form $\Omega = \Omega_2 \times \Omega_Y$ w/ (for H) $\Omega_2 = \exp(i\alpha_2^a(x)\tau^a)$ and $\Omega_1 = \exp(i\alpha_Y(x)/2)$ so that the gauge fields W_μ^a and B_μ (the hypercharge gauge boson) transform independently:

$$W_\mu^a \tau^a \to \Omega_2 W_\mu^a \tau^a \Omega_2 + i\Omega_2 \partial_\mu \Omega_2^{-1}$$

$$B_\mu \to B_\mu + \partial_\mu \alpha_Y$$

Therefore, a $U(1)_{em}$ transformation acts as

$$\Omega_Q = \exp(i\alpha^3\tau^3 + i\alpha_Y/2\,\mathbb{1}) = \exp(i\alpha_Q Q)$$

$\Rightarrow$ Transformation acts on W's same as the $U(1)$ subgroup of $SU(2)$. So we can write

$$W^\mu = \frac{1}{2}\begin{pmatrix} W_\mu^3 & \sqrt{2}W_\mu^+ \\ \sqrt{2}W_\mu^- & -W_\mu^3 \end{pmatrix}$$

w/ $Q(W^{+/-}) = \pm 1$ and $Q(W^3) = 0$.

We also have

$$W_\mu^3 \to W_\mu^3 + \partial_\mu \alpha_Q \quad \text{and} \quad B_\mu \to B_\mu + \partial_\mu \alpha_Q$$

under $U(1)_Q$. This tells us that the linear combination $Z_\mu = \underset{\uparrow \text{ some const}}{c}\left(W_\mu^3 - B_\mu\right)$ does <u>not</u> transform under $U(1)_Q$.

Since Z_μ is gauge invariant, its mass term $Z_\mu Z^\mu$ is also gauge invariant $\Rightarrow$ candidate massive neutral boson. The orthogonal linear combination does transform, and therefore can not have a mass.

Let us verify this by direct calculation. Work in unitary gauge (so that 3 Goldstone bosons are eaten). This corresponds to

$$H(x) = \begin{pmatrix} 0 \\ \frac{v + h(x)}{\sqrt{2}} \end{pmatrix}$$

w/ $h(x)$ a real scalar field, the "Higgs boson".

The covariant derivative in unitary gauge is

$$D_\mu H = \partial_\mu H - i\left(W_\mu + \tfrac{1}{2} B_\mu \mathbb{1}\right) H = \begin{pmatrix} 0 \\ \frac{\partial_\mu h}{\sqrt{2}} \end{pmatrix} - i\,\frac{v+h}{\sqrt{2}} \begin{pmatrix} \sqrt{2}\, W^+ \\ \underbrace{(B_\mu - W_\mu^3)}_{\frac{1}{c} Z_\mu} \end{pmatrix}$$

Then $|D_\mu H|^2$ will yield a mass for $W^{+/-}$ and Z. To compute these masses, recall that we have been working with the normalization

$$\mathcal{L} \supset -\frac{1}{2g^2} \mathrm{Tr}\left[W_{\mu\nu} W^{\mu\nu}\right] - \frac{1}{4g'^2} B_{\mu\nu} B^{\mu\nu}$$

To work with canonical kinetic terms, we rescale $W \to g W$ and $B \to g' B$

Now we can determine normalization for Z_μ by requiring that it has a canonical kinetic term: $Z = c\left(g W^3 - g' B\right)$

and $\mathcal{L} \supset -\frac{1}{4}\left[(W^3)^2 + B^2\right]$

$\Rightarrow$ Normalize so that Z is rotation in W^3, B space

$$Z \equiv \cos\theta_W \, W^3 - \sin\theta_W \, B = \frac{1}{\sqrt{g^2+g'^2}}\left(g W^3 - g' B\right)$$

or $\cos\theta_W \equiv c_W = \dfrac{g}{\sqrt{g^2+g'^2}}$ and $\sin\theta_W \equiv s_W = \dfrac{g'}{\sqrt{g^2+g'^2}}$

θ_W : weak mixing angle (or Weinberg angle)

The orthogonal combo is the photon:

$$A_\mu = c_W B_\mu + s_W W^3_\mu$$

We can also relate the $U(1)_Q$ coupling **4.14**

to g & g' since w/ canonical norm, we have

$A_\mu \to A_\mu + \frac{1}{e}\partial_\mu \alpha$ under $U(1)_Q$ gauge trans.

and for $U(1)_W$ and $U(1)_Y$ we have

$$W^3_\mu \to W^3_\mu + \frac{1}{g}\partial_\mu\alpha \quad \& \quad B_\mu \to B_\mu + \frac{1}{g'}\partial_\mu\alpha$$

$$\Rightarrow A_\mu = c_W B_\mu + s_W W^3_\mu \to A_\mu + \left(\frac{c_W}{g'} + \frac{s_W}{g}\right)\partial_\mu\alpha$$

$$\Rightarrow \frac{1}{e} = \frac{1}{\sqrt{g^2+g'^2}}\left(\frac{g}{g'} + \frac{g'}{g}\right) = \frac{\sqrt{g^2+g'^2}}{gg'} = \sqrt{\frac{1}{g^2}+\frac{1}{g'^2}}$$

$$\Rightarrow e = \frac{gg'}{\sqrt{g^2+g'^2}}$$

Now we can compute the masses and interactions

for the bosonic electroweak sector:

Note $D_\mu H = \begin{pmatrix} 0 \\ \frac{\partial_\mu h}{\sqrt{2}} \end{pmatrix} - i\frac{v+h}{\sqrt{2}}\begin{pmatrix} \sqrt{2}\, g\, W^+_\mu \\ \sqrt{g^2+g'^2}\, Z_\mu \end{pmatrix}$

$\Rightarrow$ Higgs kinetic terms $\Rightarrow$

$$|D_\mu H|^2 = \frac{1}{2}(\partial_\mu h)^2 + \frac{(v+h)^2}{8}\left[2g^2|W|^2 + (g^2+g'^2)Z^2\right]$$

$$\Rightarrow m_W = \frac{1}{2}gv \quad \& \quad m_Z = \frac{1}{2}\sqrt{g^2+g'^2}\,v = \frac{1}{2}\frac{g}{c_W}v = \frac{m_W}{c_W}$$

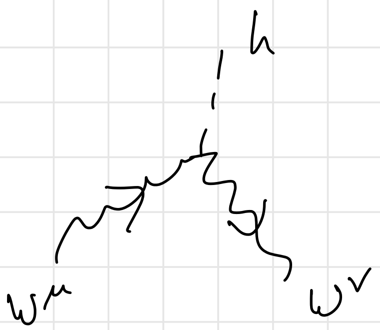
Note, the combination

$$\rho \equiv \frac{m_W^2}{m_Z^2 c_W^2} = 1$$

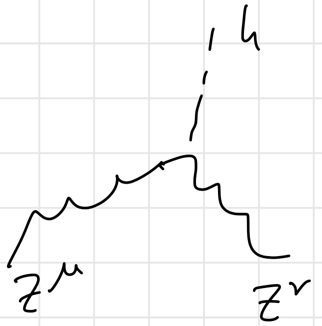
tree-level

This is an important precision observable for testing the SM (sensitive to loop effects and BSM physics).

The Higgs kinetic term also gives us interactions:



$$= \frac{i}{2} g^2 v \eta_{\mu\nu} = 2i \frac{m_W^2}{v} \eta_{\mu\nu}$$



$$= \frac{i}{2} (g^2 + g'^2) v \eta_{\mu\nu} = 2i \frac{m_Z^2}{v} \eta_{\mu\nu}$$

$$= \frac{i}{2} g^2 \eta_{\mu\nu}$$

$$= \frac{i}{2} \frac{g^2}{c_W^2} \eta_{\mu\nu}$$

The Higgs potential gives

$$V(H) \overset{\substack{\text{unitary}\\\text{gauge}}}{=} \frac{1}{2}(2\lambda v^2)h^2 + \lambda v h^3 + \frac{\lambda}{4}h^4$$

$$\Rightarrow m_h = \sqrt{2\lambda}\, v = \sqrt{2}\mu$$

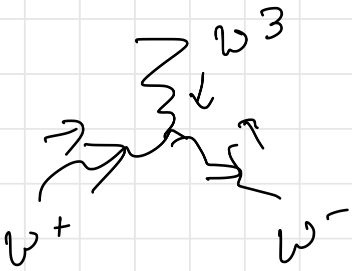
and we have Higgs self interactions

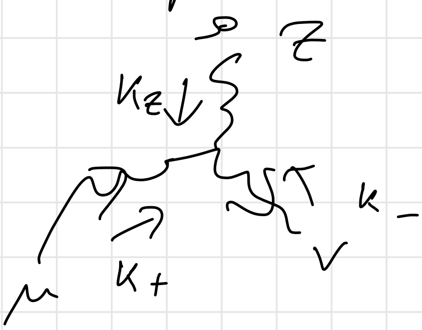
 $= -6i\lambda v$  $= -6i\lambda$

We also have gauge boson self interactions:
All gauge self interactions come from $SU(2)$ structure $\Rightarrow$ We can use the calculations for pure $SU(2)$, with the replacement

$$W^3 = c_w Z + s_w A \quad (\text{also } B = c_w A - s_w Z)$$

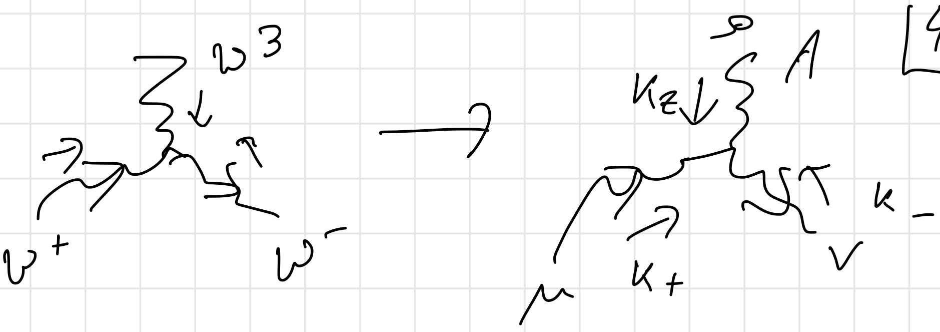
where A is the photon. $\Rightarrow$



$\longrightarrow$


$$= -ig c_w \left[(k_+ - k_-)_\rho \eta_{\mu\nu} + (k_- - k_Z)_\mu \eta_{\nu\rho} + (k_Z - k_+)_\nu \eta_{\mu\rho} \right]$$

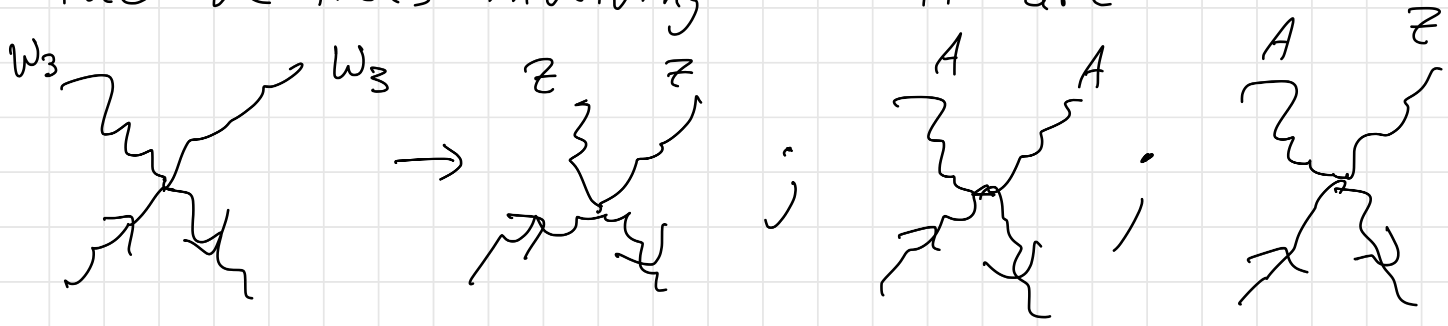
Also, we have:



$= -i g s_W [\dots]$

Similarly, the pure $W^{+/-}$ vertices are unchanged

The vertices involving Z & A are



$\sim g^2 c_W^2$; $\sim g^2 s_W^2$; $\sim g s_W c_W$

Parameters of bosonic sector of SM

We see that we have 4 parameters:

g, g', μ, λ

Historically, what was actually measured first were

- e from electromagnitism / QED
- v (really G_F) from weak decays (μ decay)
- s_W from $\nu - e$ scattering and Z-pole
- λ Higgs mass

Summary for bosonic sector

$$m_W = \frac{1}{2} g v = c_W m_Z$$

$$m_h = \sqrt{2\lambda}\, v$$

$$s_W = \frac{g'}{\sqrt{g^2+g'^2}} \qquad c_W = \frac{g}{\sqrt{g^2+g'^2}}$$

$$e = \frac{g g'}{\sqrt{g^2+g'^2}} = g s_W = g' c_W$$

Measured values:

$m_W \simeq 80$ GeV, $m_Z = 91$ GeV, $m_h = 125$ GeV

$v \simeq 246$ GeV, $s_W^2 \simeq 0.23$, $g \simeq 0.65$,

$g' \simeq 0.35$, $\lambda \simeq 0.13$, $\mu \simeq 88$ GeV

Quarks and Leptons

Now we need to introduce the spin 1/2 matter fields. There are two types: the quarks (charged under $SU(3)$) and the leptons (neutral under $SU(3)$). There are three copies of each, which are called the "families".

In more detail, a family of <u>leptons</u> consists of 3 Weyl spinors. We will use Dirac notation, $\psi_{L,R}$ with the property

$$\gamma_5 \psi_L = -\psi_L \quad , \quad \gamma_5 \psi_R = +\psi_R$$

The 3 fields are l_L, ν_L, l_R

It is possible that ν_R also exists, but we do not know for sure so we do not typically include it in the SM.

In other words, neutrinos are massless in the SM. Notice we treat left and right differently.

We combine the left handed fields into the "lepton doublet" $L_L = \begin{pmatrix} \nu_L \\ l_L \end{pmatrix}$

which transforms in the fundamental rep of $SU(2)$. We want l_L to be the electron, so we should choose its hypercharge so that $Q(l_L) = -1$. Using $Q = T^3 + Y$ and $T^3 = -\frac{1}{2}$ for l_L we need $Y(l_L) = -1/2$

This implies $Q(\nu) = \frac{1}{2} - \frac{1}{2} = 0$ so the neutrinos are neutral.

We also need l_R to have $Q(l_R) = -1$. Since it is an $SU(2)$ singlet, we have $T_3(l_R) = 0 \Rightarrow Y(l_R) = -1$

$$\Rightarrow L_L = \begin{pmatrix} \nu_L \\ l_L \end{pmatrix} \in \underset{\uparrow\, SU(2)}{\mathbf{2}}{}_{\underset{\nwarrow\, U(1)_Y}{-1/2}} \quad \& \quad l_R \in \underline{\mathbb{1}}_{-1}$$

When we write Dirac spinors, $l = l_L + l_R$, we also write $\nu = \nu_L + \nu_R$. But ν_R does not couple to anything so it plays no physical role.

Quarks

One family of quarks is made up of

$$Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix} \in \mathbf{2}_{1/6} \;\&\; u_R \in \underline{\mathbb{1}}_{2/3} \;\&\; d_R \in \underline{\mathbb{1}}_{-1/3}$$

(neglecting $SU(3)$)

Families

The families have identical interaction structure. Only difference is the masses and mixing effects (we will discuss this later when we introduce the CKM matrix).

The 3 families are:

	I		II		III	
leptons	ν_e	(0)	ν_μ	(0)	ν_τ	(0)
	e	$(511\ \text{keV})$	μ	$(106\ \text{MeV})$	τ	$(1.8\ \text{GeV})$
quarks	u	$(2.4\ \text{MeV})$	c	$(1.3\ \text{GeV})$	t	$(172\ \text{GeV})$
	d	$(4.8\ \text{MeV})$	s	$(104\ \text{MeV})$	b	$(4.2\ \text{GeV})$

Let's work out their couplings to the $W^{+/-}$, Z, A:

Writing $W_\mu = \frac{1}{2}\begin{pmatrix} W_\mu^3 & \sqrt{2}\, W_\mu^+ \\ \sqrt{2}\, W_\mu^- & -W_\mu^3 \end{pmatrix}$

Then for a generic doublet D_L we have

$$D_\mu D_L = \left(\partial_\mu \mathbb{1} - ig W_\mu - ig' Y(D_L) B_\mu \mathbb{1} \right) D_L$$

For a generic singlet S_R we have

$$D_\mu S_R = \left(\partial_\mu - ig' Y(S_R) B_\mu \right) S_R$$

This gives "charged current" interactions for $D_L = \begin{pmatrix} \psi_{u,L} \\ \psi_{d,L} \end{pmatrix}$

$$\mathcal{L} \supset \frac{1}{\sqrt{2}}\, g\, \overline{\psi}_{u,L} \gamma^\mu W_\mu^+ \psi_{d,L} + \frac{1}{\sqrt{2}}\, g\, \overline{\psi}_{d,L} \gamma^\mu W_\mu^- \psi_{u,L}$$

If we want to express this using $\psi = \psi_L + \psi_R$

we use $\psi_{L/R} = \dfrac{1 \mp \gamma^5}{2} \psi$

$$\Rightarrow \mathcal{L} = \frac{g}{2\sqrt{2}} \overline{\psi}_u \gamma^\mu (1-\gamma^5) W_\mu^+ \psi_d$$

$$+ \frac{g}{2\sqrt{2}} \overline{\psi}_d \gamma^\mu (1-\gamma^5) W_\mu^- \psi_u$$

$\Rightarrow$

$$= i \frac{g}{2\sqrt{2}} \gamma^\mu (1-\gamma^5)$$

ψ_d $\quad$ ψ_u

l or d $\quad$ ν or u

$$= i \frac{g}{2\sqrt{2}} \gamma^\mu (1-\gamma^5)$$

ψ_u $\quad$ ψ_d

Note the universality of the couplings.

We can write this in terms of a "charged current"

$$J_\mu^+ \equiv \sum_{\text{doublets}}' \bar{\psi}_d \gamma_\mu (1-\gamma^5) \psi_u$$

$$= \bar{l} \gamma_\mu (1-\gamma^5) \nu + \bar{d} \gamma_\mu (1-\gamma^5) u + \text{other families}$$

and $J_\mu^- = (J_\mu^+)^\dagger$

$$\Rightarrow \mathcal{L} \supset \frac{g}{2\sqrt{2}} W_\mu^+ J^{-\mu} + \frac{g}{2\sqrt{2}} W_\mu^- J^{+\mu}$$

"Neutral currents" (Z + A) follow from the same logic.

$$\mathcal{L}_{D_L} \supset \sum_{l=u,d} \bar{\psi}_{l,L} \gamma^\mu \left(g T^3(l) W_\mu^3 + g' Y B_\mu \right) \psi_{l,L}$$

Then using $Q = T^3 + Y$

$$\Rightarrow \mathcal{L}_{D_L} \supset \sum_l' \bar{\psi}_{l,L} \gamma^\mu \left[(g W^3 - g' B) T^3(l) + g' Q(l) B_\mu \right] \psi_{l,L}$$

Plug in $W^3 = c_w Z + s_w A$ & $B = c_w A - s_w Z$

$$\Rightarrow \mathcal{L}_{D_L} \supset \sum_l' e\, Q(l)\, \bar{\psi}_{L,l} \gamma^\mu A_\mu \psi_{l,L} = \mathcal{L}_{QED}$$

For the Z, we have

$$\mathcal{L}_{D_L} \supset \frac{g}{c_W} \sum_{\ell} \bar{\psi}_{\ell,L} \gamma^\mu Z_\mu \left[T^3(\ell) - s_W^2 Q(\ell) \right] \psi_{\ell,L}$$

For right handed fields, we have

$$\mathcal{L}_{S_R} = g' B_\mu Y(\psi_R) \bar{\psi}_R \gamma^\mu \psi_R$$

$$= e Q(\psi_R) \bar{\psi}_R \gamma^\mu A_\mu \psi_R - \frac{g}{c_W} s_W^2 Q(\psi_R) \bar{\psi}_R \gamma^\mu Z_\mu \psi_R$$

$\Rightarrow$ Again we get QED.

We can write the coupling to the Z in terms of a neutral current

$$J_\mu^0 \equiv \sum_{\substack{\text{all} \\ \text{fermions}}} \bar{\psi}_i \gamma^\mu \left(g_V(i) - g_A(i) \gamma^5 \right) \psi_i$$

where $\mathcal{L} = \frac{g}{c_W} Z_\mu J^{0\mu}$ and we have defined "vector" and "axial" couplings

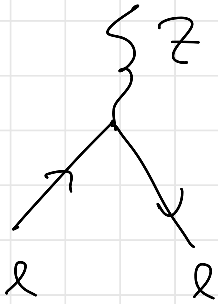
$$g_V(i) = \frac{1}{2} \left[T^3(i) - 2 s_W^2 Q(i) \right]$$

$$g_A(i) = \frac{1}{2} \left[T^3(i) \right]$$

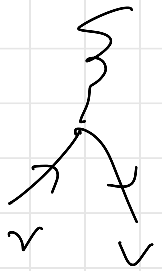
where T^3 denotes the T^3 quantum number for the left handed component of the Dirac field.

Feynman rules:

$= ieQ\gamma^\mu$



$= i\frac{g}{4c_w}\gamma^\mu\left[(-1+4s_w^2)-\gamma^5\right]$



$= i\frac{g}{4c_w}\gamma^\mu(1-\gamma^5)$

w/ similar rules for the quarks.

Global Symmetries of the SM

As of now, we note that there is an $U(3)_L\times U(3)_e\times U(3)_Q\times U(3)_u\times U(3)_d$ global symmetry of the SM. We can see this by organizing the families into a multiplet:

$$Q_i = \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix} \qquad L_i = \begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} \quad \text{etc.}$$

Then $U(3)_L$ rephases Q_i & L_i while $U(3)_e$ rephases e_i, d_i, u_i.

When we introduce Yukawa couplings to give mass to the fermions, this will be broken to $U(1)_B\times U(1)_\ell$

Now that we have the Lagrangian and Feynman rules, it will be useful to understand the properties of the theory under the discrete transformations Parity P and Charge conjugation C.

Parity

Parity (and time reversal) are transformations of the full Lorentz group:

$$\underline{P}: (t, \vec{x}) \rightarrow (t, -\vec{x})$$

$$T: (t, \vec{x}) \rightarrow (-t, \vec{x})$$

We will focus on P here. Note that by the CPT theorem (unitary, Lorentz invariant Lagrangians are invariant under CPT) we only need to consider P, C, and CP.

We will show that some of the weak interactions break C and P but preserve CP.

Then, when we introduce the masses for the fermions, we will find a CP violating effect.

We are looking for representations of the $\hat{P}$ operator. 4.27

For real scalar fields, $\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 - \frac{m}{2}\varphi^2$

So $P^2 = 1 \Rightarrow \varphi(t,\vec{x}) \xrightarrow{P} \pm\varphi(t,-\vec{x})$

we call "+" scalars and "−" pseudoscalars

For complex scalars, there is a subtelty that we can always rephase them since $\varphi \to \pm\varphi$ is subgroup of $U(1)$. As long as we fix a phase convention, we can identify $\varphi(t,\vec{x}) \xrightarrow{P} \pm\varphi(t,-\vec{x})$ for complex scalars.

Note that $\partial_\mu \varphi(t,\vec{x}) \xrightarrow{P} P_\mu^{\ \nu}\partial_\nu\varphi(t,-\vec{x})$

w/ $P_\mu^{\ \nu} = \mathrm{diag}(+,-,-,-)$

So derivatives transform as a 4-vector as they must.

The Higgs is a "Scalar" in the SM:

$$H(t,\vec{x}) \xrightarrow{P} H(t,-\vec{x})$$

Next, let's discuss gauge bosons. Again we have two choices: $A_\mu(t,-\vec{x}) \xrightarrow{P} \pm P_\mu^{\ \nu} A_\mu(t,-\vec{x})$

"+" is a vector and "−" is a pseudovector.

The field strength is invariant for both choices: $F_{\mu\nu} \xrightarrow{P} (\pm)^2 P_\mu{}^\lambda P_\nu{}^\rho F_{\lambda\rho}(t,-\vec{x})$

To determine the parity of the gauge bosons, we can appeal to the covariant derivative

$D_\mu = \partial_\mu - ie A_\mu$. So we need A_μ to transform like $\partial_\mu \Rightarrow$ gauge bosons are "vectors" under P.

Note that $\varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \xrightarrow{P} -\varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$

This term has no impact in perturbation theory (it is a "surface term") so the gauge sector preserves P.

$\Rightarrow$ The bosonic sector of the SM preserves P

<u>Fermions</u>: You know from QFT that Dirac fermions transform as $\psi(t,\vec{x}) \xrightarrow{P} \gamma^0 \psi(t,-\vec{x})$

$\Rightarrow \bar{\psi}_1 \gamma^\mu \psi_2 \xrightarrow{P} \bar{\psi}_1 \gamma^0 \gamma^\mu \gamma^0 \psi_2 = P^\mu{}_\nu \bar{\psi} \gamma^\mu \psi_2$

$\Rightarrow$ "vector current"

and $\bar{\psi}_1 \gamma^\mu \gamma^5 \psi_2 \xrightarrow{P} - P^\mu{}_\nu \bar{\psi}_1 \gamma^\mu \gamma^5 \psi_2$

$\Rightarrow$ "axial current"

P is a symmetry of QED because the photon couples to a vector current.

In the SM we have couplings to both types of currents. Recall the charged current:

$$J_\mu^+ \equiv \sum_{\text{doublets}}' \bar{\psi}_d \gamma_\mu (1-\gamma^5) \psi_u$$

$$= \bar{l} \gamma_\mu (1-\gamma^5) \nu + \bar{d} \gamma_\mu (1-\gamma^5) u + \text{other families}$$

and the neutral current:

$$J_\mu^0 \equiv \sum_{\substack{\text{all} \\ \text{fermions}}}' \bar{\psi}_i \gamma^\mu (g_V(i) - g_A(i)\gamma^5) \psi_i$$

So P is <u>not</u> a fundamental symmetry of the SM, aka the SM is "chiral".

It is only an "accidental" symmetry of QED that emerges at low energies.

<u>Charge Conjugation</u>

<u>Scalars</u> $\varphi(x) \xrightarrow{C} \varphi^*(x)$

This will be true for the Higgs $H \xrightarrow{C} H^*$

(Note there is a subtelty w/ choosing the correct notion of charge for non-Abelian groups, revisit later)

For the photon, we choose $A_\mu \xrightarrow{C} -A_\mu$.

One way to see this is we want

$D_\mu \varphi \xrightarrow{C} (D_\mu \varphi)^*$ for a charged scalar field

$$\Rightarrow (\partial_\mu - ieA_\mu)\phi \xrightarrow{C} (\partial_\mu - ie(-A_\mu))\varphi^* = (D_\mu \varphi)^*$$

For non-Abelian, we want a similar notion.

We can use the fact that any compact Lie group admits an "automorphism" of the algebra that we will call "C" such that

$C^a{}_b = \mathrm{diag}(\pm 1 \ldots \pm 1)$ that leaves the structure constants invariant:

$$C^a{}_{a'} C^b{}_{b'} C^c{}_{c'} f^{a'b'c'} = f^{abc}$$

We can use C to define a new set of generators $T_C^a = C^a{}_{a'} T^{a'}$ since

$$[T_C^a, T_C^b] = if^{abc} T_C^c$$

If we act with C on the gauge fields,

$$A_\mu^a(x) \xrightarrow{C} C^a{}_{a'} A_\mu^{a'}(x)$$

then the F^2 term is invariant:

$$F_{\mu\nu}^{a} \xrightarrow{G} \mathcal{C}^{a}{}_{a'} F_{\mu\nu}^{a'}$$

$$\Rightarrow F_{\mu\nu}^{a} F^{\mu\nu,a} \xrightarrow{G} \mathcal{C}^{a}{}_{a'} \mathcal{C}^{a}{}_{b'} F_{\mu\nu}^{a'} F^{\mu\nu,b'}$$

$$= F_{\mu\nu}^{a} F^{\mu\nu a}$$

since $(\mathcal{C})^2 = 1$.

Same for $\varepsilon F F$ term.

$\Rightarrow$ Gauge self interactions respect G

($\Rightarrow$ $\varepsilon F F$ term breaks CP $\Rightarrow$ "strong CP problem")

[See page 17 of Wulzer Notes 10 or Georgi group theory book for general proof for construction of $\mathcal{C}$]

For the fundamental representation of $SU(2)$, $\mathcal{C}$ is given by (essentially determined by complex conjugation: $-(t_c^{a})^* = \mathcal{C}^{a}{}_{a'} t_c^{a'}$

$$\Rightarrow -(\tau^{1,3})^* = -\tau^{1,3} \; ; \; -(\tau^{2})^* = +\tau^{2}$$

$\Rightarrow$ Can determine action of $\mathcal{C}$:

$$\Rightarrow W_\mu^{1,3} \xrightarrow{G} -W_\mu^{1,3} \; ; \; W_\mu^{2} \xrightarrow{G} +W_\mu^{2}$$

$$\Rightarrow W_\mu^{+/-} \xrightarrow{G} -W_\mu^{-/+} = (-W^{+/-})^*$$

Since $U(1)_Y$ is Abelian, we have

$$B_\mu \xrightarrow{C} -B_\mu \quad \Rightarrow \quad A_\mu \xrightarrow{C} -A_\mu$$
$$Z_\mu \xrightarrow{C} -Z_\mu$$

We can summarize the C transformation rule as:

$$A_\mu = A_\mu^a t^a \xrightarrow{C} -A_\mu^* = -A_\mu^t$$

$$\Rightarrow F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] \Rightarrow -F_{\mu\nu}^* = -F_{\mu\nu}^t$$

$$\Rightarrow Tr[F^2] \xrightarrow{C} Tr[F^2]$$

Now that we have $\mathcal{E}$, we can revisit the rule for the Higgs: $(H^a)^* = \mathcal{E}^a{}_{a'} H^{a'}$

$$\Rightarrow H^a \xrightarrow{C} (H^a)^*$$

$$\Rightarrow D_\mu H = \partial_\mu H - i W_\mu H - i \tfrac{1}{2} B_\mu H$$
$$\xrightarrow{C} \partial_\mu H^* + i W_\mu^* H^* + i \tfrac{1}{2} B_\mu H^* = (D_\mu H)^*$$

$$\Rightarrow \mathcal{L}_{Higgs} = |D_\mu H|^2 - V(|H|^2) \xrightarrow{C} \mathcal{L}_{Higgs}$$

$\Rightarrow$ Bosonic sector of the SM is C invariant.

Note: if we work in a general gauge w/

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} -i\sqrt{2}\pi^+ \\ v + h + i\pi^0 \end{pmatrix} \Rightarrow \quad \pi^+ \xrightarrow{C} -\pi^- = -(\pi^+)^* \; ; \; \pi^0 \xrightarrow{C} \pi^0$$
$$h \xrightarrow{C} h$$

$\Rightarrow$ Physical Higgs is CP even.

Eaten components are CP odd and transform as they need to be consistent w/ being longitudinal modes of $W^{+/-}$ + Z^0.

(see Wulzer for discussion of subtelties w/ defining C for triplet Higgs)

Fermions

For Dirac fermions, we have

$$\psi \xrightarrow{C} \mathcal{C}\bar{\psi}^t \quad ; \quad \bar{\psi} \xrightarrow{C} \psi^t \mathcal{C}$$

w/ $\mathcal{C} = i\gamma^2\gamma^0$ s.t. $\mathcal{C}^* = \mathcal{C}$ and $\mathcal{C}^2 = -\mathbb{1}$

Also have $\mathcal{C}\gamma^\mu\mathcal{C} = (\gamma^\mu)^t$ and $[\mathcal{C}, \gamma^5] = 0$

$$\Rightarrow \bar{\psi}_1\gamma^\mu\psi_2 \xrightarrow{C} \psi_1^t\,\mathcal{C}\gamma^\mu\mathcal{C}\,\bar{\psi}_2^t = \psi_1^t(\gamma^\mu)^t\bar{\psi}_2^t$$

$$= -\bar{\psi}_2\gamma^\mu\psi_1$$

anti-commute

$$\text{and } \bar{\psi}_1\gamma^\mu\gamma^5\psi_2 \xrightarrow{C} \psi_1^t\,\mathcal{C}\gamma^\mu(-\mathcal{C}^2)\gamma^5\mathcal{C}\,\bar{\psi}_2^t$$

$$= \psi_1^t(\gamma^\mu)^t\gamma^5(-\mathcal{C}^2)\bar{\psi}_2^t$$

$$= -\bar{\psi}_2\gamma^5\gamma^\mu\psi_1 = +\bar{\psi}_2\gamma^\mu\gamma^5\psi_1$$

So using $\mathcal{L}_{cc} = W_\mu^+ J^{\mu -} + h.c.$

$\Rightarrow$ vector coupling preserves charge while axial coupling does not

and $\mathcal{L}_{nc} = Z_\mu^0 J^{\mu 0} \Rightarrow$ Same conclusion.

$\Rightarrow$ axial SM gauge interactions with fermions break C

$\Rightarrow$ C and P both broken by axial couplings.

<u>But</u> CP is conserved by these couplings, for now...

Fermion Masses and Mixings

There is one more set of terms we can write in the Lagrangian with $D \leq 4$, The "Yukawa couplings": $\mathcal{L} \supset y_u H Q u + y_d H Q d + y_e H L e$

Let's understand their structure and implications in detail.

Note: $SU(3)_c$ forbids any Yukawa coupling between quarks and leptons.

Lepton Yukawas

Must be $SU(2)_L \times U(1)_Y$ invariant

Recall $L_L \in \mathbf{2}_{-1/2}$, $e_R \in \mathbb{1}_{-1}$, $H \in \mathbf{2}_{1/2}$

Lorentz invariance for spinors requires $L\bar{e}$

Note $Y(L\bar{e}) = 1/2 \Rightarrow$ Must contract w/ H:

$$(H^*)^\alpha L_{L,\alpha} = H^* \cdot L$$

$$\Rightarrow \mathcal{L} \supset -y_e \bar{e}_R H^* \cdot L_L - y_e^* \bar{L}_L \cdot H e_R$$

Plugging in the Higgs vev in unitary gauge, we have

$$H = \begin{pmatrix} 0 \\ \frac{v+h}{\sqrt{2}} \end{pmatrix} \qquad L_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$$

$$\Rightarrow \mathcal{L} \supset -\frac{y_e}{\sqrt{2}}(v+h)\bar{\ell}_R \ell_L - \frac{y_e^*}{\sqrt{2}}(v+h)\bar{\ell}_L \ell_R$$

$$\underset{\uparrow}{=} \frac{y_e}{\sqrt{2}} v \bar{\ell}\ell - \frac{y_e}{\sqrt{2}} h \bar{\ell}\ell$$

rephase ℓ_L or ℓ_R to make y_e real

$\Rightarrow$ Charged lepton has a Dirac mass $m_e = \frac{y_e v}{\sqrt{2}}$

and there is a new Feynman vertex:

$$= -i\frac{y_e}{\sqrt{2}} = -i\frac{m_e}{v}.$$

*The neutrino stays massless.

From the masses of the leptons, we can determine

$$y_e = \frac{\sqrt{2} m_e}{v} = \left(\underset{e}{3 \times 10^{-6}},\ \underset{\mu}{6 \times 10^{-4}},\ \underset{\tau}{10^{-2}} \right)$$

$\Rightarrow$ Couplings between Higgs boson and leptons is tiny and typically can be ignored.

Quark Yukawas

To write the up-type Yukawas, we will need

$$H^c_\alpha \equiv \varepsilon_{\alpha\beta} (H^*)^\beta = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} h_u^* \\ h_d^* \end{pmatrix}$$

This is a doublet of $SU(2)$ w/ hypercharge $-1/2$

It transforms like a doublet because $\varepsilon_{\alpha\beta}$ "lowers" $SU(2)$ indices. We can check explicitly.

Using $\sigma_2 \tau^{a*} \sigma_2 = -\tau^a \Rightarrow \sigma_2 e^{-i\alpha^a \tau^{*a}} \sigma_2 = e^{i\alpha^a \tau^a}$

$$\Rightarrow H^c \equiv i\sigma_2 H^* \xrightarrow{SU(2)} i\sigma_2 e^{-i\alpha^a \tau^{a*}} H^* = e^{i\alpha^a \tau^a} \underbrace{i\sigma_2 H^*}_{= H^c}$$

Then the quark Yukawas are

$$\mathcal{L} \supset - y_d \overline{Q_L} \cdot H d_R + h.c. \quad \left(Y = -\tfrac{1}{6} + \tfrac{1}{2} - \tfrac{1}{3} = 0 \right)$$

$$- y_u \overline{Q_L} \cdot H^c u_R + h.c. \quad \left(Y = -\tfrac{1}{6} - \tfrac{1}{2} + \tfrac{2}{3} = 0 \right)$$

In Unitary gauge, $H^c = \begin{pmatrix} \frac{v+h}{\sqrt{2}} \\ 0 \end{pmatrix} \Rightarrow$ gives mass to up-type quarks

$$\Rightarrow \mathcal{L} \supset -\frac{y_d}{\sqrt{2}} v \bar{d} d - \frac{y_d}{\sqrt{2}} h \bar{d} d - \frac{y_u}{\sqrt{2}} v \bar{u} u - \frac{y_u}{\sqrt{2}} h \bar{u} u$$

$$m_{u/d} = \frac{y_{u/d}}{\sqrt{2}} v$$

$$= -i \frac{y_d}{\sqrt{2}} = -i \frac{m_d}{v}$$

Yukawas are small except for the top quark:

$$m_t \simeq 173 \text{ GeV} \Rightarrow y_t \simeq 1$$

This is the largest coupling in the SM (above the QCD confinement scale)

Impact of family structure

This was correct for a 1 family model.

But the SM has 3 families $\Rightarrow$ can have mixing between the different families.

The point is that we are allowed to do field redefinitions without changing any of the physical observables.

In this case, the field redefinitons are rotations among the families and rephasing of the fields. We will see that we can remove many of these mixing parameters using such field redefinitions.

3 families of leptons

First we will show that the mixing parameters have no physical impact in the lepton sector (as long as the neutrinos are massless).

Take the most general lepton Yukawa matrix:

$$\mathcal{L} \supset -(Y_e)_i{}^j \, \overline{L}_L^i H e_{R,j} - (Y_e^\dagger)^i{}_j \, \bar{e}^j H^\dagger \cdot L_{L,i}$$

w/ $i = 1,2,3$

It seems we now have 9 complex parameters to specify. We will now show that we can absorb all but 3 real parameters into field redefs.

Note that the kinetic term and gauge interactions are diagonal:

$$\mathcal{L} \supset \sum_{i=1}^{3} \bar{L}_L^{\,i}\, i\not{D} L_{L,i} + \sum_{j=1}^{3} \bar{e}_R^{\,j}\, i\not{D} e_{R,j}$$

$\Rightarrow$ There is a $U(3)_L \times U(3)_e$ global symmetry group

$$L_{L,i} \to (V_L)_i^{\ j} L_{L,j} \qquad L_L \in (3,1)$$

$$\bar{L}_L^{\,i} \to (V_L^\dagger)^i_{\ j} \bar{L}_L^{\,j} \qquad \bar{L}_L \in (\bar{3},1)$$

$$e_{R,i} \to (V_e)_i^{\ j} e_{R,j} \qquad e_R \in (1,3)$$

$$\bar{e}_R^{\,i} \to (V_e^\dagger)^i_{\ j} \bar{e}_R^{\,j} \qquad \bar{e}_R \in (1,\bar{3})$$

where $V^\dagger V = 1$ (the V's are unitary)

Performing this transformation does not change the kinetic and gauge terms. It will change the Yukawa terms. We can always write

$$Y_e = U_L Y_e^D U_R^\dagger \quad \text{w/} \quad U_{L,R}^\dagger U_{L,R} = 1$$

by doing a Singular Value Decomposition.

w/
$$Y_e^D = \begin{pmatrix} Y_e & & 0 \\ 0 & Y_\mu & \\ & & Y_\tau \end{pmatrix}$$ is diagonal and has positive eigenvalues

$\Rightarrow$ We can write (supressing all the indices)

$$\mathcal{L} \supset -\bar{\vec{L}}_L H U_L Y_e^D U_e \vec{e}_R + h.c.$$

Then we can perform a $U(3)_L \times U(3)_e$ transformation $V_{L,e} = U_{L,R}$

$$\Rightarrow \mathcal{L} \supset -\bar{\vec{L}} H Y_e^D \vec{e}_R + h.c.$$

This has profound implications. It tells us that the most general $\mathcal{L}$ with operators $d \leq 4$ does not allow lepton flavor transitions.

There is an exact $U(1)$ symmetry left over after rotating to the diagonal basis:

$$L_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \rightarrow e^{i\alpha_\ell} \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$$

$$\bar{e}_R \rightarrow e^{-i\alpha_\ell} \bar{e}_R \qquad \Rightarrow U(1)_\ell \quad \text{"lepton number"}$$

There is one for each family $\Rightarrow$ 3 conserved charges "electron number", "muon number" and "tau number"

These are "accidental symmetries" of the SM.

They can be violated by $d > 4$ operators.

So processes like $\mu \to e\gamma$, $\mu \to eee$, ...
are forbidden in the SM.

Quark masses and mixings

Now we have

$$\mathcal{L} \supset \sum_{i=1}^{3} \overline{Q}_L^{\,i}\, i\not{D}\, Q_{L,i} + \sum_{j=1}^{3} \overline{u}_R^{\,j}\, i\not{D}\, u_{R,j} + \sum_{k=1}^{3} \overline{d}_R^{\,k}\, i\not{D}\, d_{R,k}$$

w/ flavor group $U(3)_Q \times U(3)_u \times U(3)_d$

$\Rightarrow$ Can remove 3 unitary matrices worth of parameters in the Yukawa couplings.

The new feature is we would need 4 independent unitary matrices to diagonalize both Yukawa couplings in general, i.e., u_L and d_L are linked.

Let us try to do the same trick:

$$Y_u = U_{L,u} Y_u^D U_{R,u}^\dagger \quad ; \quad Y_d = U_{L,d}\, Y_d^D U_{R,d}^\dagger$$

Set $V_Q = U_{L,u}$, $V_u = U_{R,u}$, $V_d = U_{R,d}$

$$\Rightarrow \mathcal{L} \supset -\overline{\vec{Q}}_L H^c Y_u^D \vec{u}_R - \overline{\vec{Q}}_L V_Q^\dagger U_{L,d} Y_d^D \vec{d}_R + h.c.$$

So we have the 6 real parameters

$$Y_u = \mathrm{diag}(Y_u, Y_c, Y_t)$$

$$Y_d = \mathrm{diag}(Y_d, Y_s, Y_b)$$

and a unitary matrix worth of parameters

$$V_{CKM} \equiv V_Q^\dagger \cdot U_{L,d} = U_{L,u}^\dagger \cdot U_{L,d}$$

"Cabibbo - Kobayashi - Maskawa" matrix

$\Rightarrow$ There is no exact notion of quark flavor in the SM. So the 3 would be $U(1)^3$ global symmetry is broken down to a single $U(1)_B$ "Baryon number".

The CKM matrix has another important impact: it can break CP symmetry. To see this, we need to understand the CP structure of the Yukawa couplings. We know:

$$\psi \xrightarrow{P} \gamma^0 \psi \quad , \quad H \xrightarrow{P} H \quad , \quad H^c \xrightarrow{P} H^c$$

$$\Rightarrow \quad \bar{\psi}_1 H \left(\frac{1-\gamma^5}{2}\right) \psi_2 \xrightarrow{P} \bar{\psi}_1 H \left(\frac{1+\gamma^5}{2}\right) \psi_2 \qquad \left(\text{used } \{\gamma^\mu, \gamma^5\} = 0\right)$$

Under C: $\quad \psi \to i\gamma^2\gamma^0 \bar{\psi}^t \; ; \; H \to H^*$

$$\Rightarrow \bar{\psi}_1 H\left(\frac{1+\gamma^5}{2}\right)\psi_2 \xrightarrow{CP} \psi_1^t H^* (i\gamma^2\gamma^0)^2\left(\frac{1-\gamma^5}{2}\right)\bar{\psi}_2^t$$

$$\left[(i\gamma^2\gamma^0)^2 = \mathbb{1}\right] \qquad = -\psi_1^t H^*\left(\frac{1-\gamma^5}{2}\right)\bar{\psi}_2^t$$

$$\text{anti-commute} = +\bar{\psi}_2 H^*\left(\frac{1-\gamma^5}{2}\right)\psi_1$$

$$\left[(\bar{\psi}_1\gamma^5\psi_2)^\dagger = -\bar{\psi}_2\gamma^5\psi_1\right] = \left(\bar{\psi}_1 H\left(\frac{1+\gamma^5}{2}\right)\psi_2\right)^\dagger$$

$\Rightarrow$ Yukawa operator $\mathcal{O}_Y$ transforms to its conjugate under CP.

$$\Rightarrow \mathcal{L}_Y = Y\mathcal{O}_Y + Y^*\mathcal{O}_Y^\dagger \xrightarrow{CP} Y\mathcal{O}_Y^\dagger + Y^*\mathcal{O}_Y$$

$$= \mathcal{L}_Y \quad \text{iff} \quad Y = Y^*$$

So complex Yukawa couplings break CP.

In our basis above, we diagonalized the up-type Yukawas using SVD. So the up-type sector preserves CP.

The down-type sector depends on V_{CKM}.

Since Y_d^D is real, the down-type sector preserves CP if $V_{CKM}^* = V_{CKM}$.

Since V_{CKM} is unitary, it in principle has many complex entries. <u>But</u> we did not exhaust our field redefinition freedom yet, i.e., some of the V_{CKM} parameters are unphysical.

Our Lagrangian is invariant under 3 $U(1)$ phase rotations in the up-sector (just like for the leptons):

$$Q_L^j \to e^{i\alpha_j} Q_L^j \;;\; u_R^j \to e^{i\alpha_j} u_R^j \quad j=1,2,3$$

We can also rephase all the d_R's:

$$d_R^j \to e^{i\beta_j} d_R^j \quad w/\ j=1,2,3$$

The down-type Yukawas break all 6 of these $U(1)$ global symmetries down to $U(1)$ Baryon number: $\alpha_j = \beta_j \equiv \alpha_B \Rightarrow$ There are 5 phase rotations we can do to remove phases in V_{CKM}:

$$\mathcal{L} \supset -\vec{\bar{Q}}_L H \cdot (V_{CKM} \cdot Y_d^D)\, \vec{d}_R + h.c.$$

both diagonal so they commute

$$\to -\vec{\bar{Q}}_L H \cdot \left(e^{-i\,\mathrm{diag}(\vec{\alpha})}\, V_{CKM}\, e^{i\,\mathrm{diag}(\vec{\beta})}\, Y_d^D\right) \vec{d}_R + h.c.$$

So we can redefine

$$V_{CKM} \to e^{-i\,\mathrm{diag}(\vec{\alpha})} V_{CKM}\, e^{i\,\mathrm{diag}(\vec{\beta})}$$

Taking $\alpha_i = \beta_i = \alpha_B$ has no impact, so this is how we remove 5 phases.

Let us count parameters:

We know that an $N \times N$ unitary matrix has N^2 real parameters. Of these,

$N(N-1)/2$ are angles and

$N(N+1)/2$ are phases.

- Assume $N=1$ generations $\Rightarrow$ 1 phase

  Can absorb this phase w/ chiral phase rotation $\alpha = -\beta$ (confirms what we already knew for $N=1$ model). $\alpha = \beta$ rotation does nothing.

- Assume $N=2$ generations $\Rightarrow$ 1 angle + 3 phases.

  Now we have 2 α's and 2 β's that can absorb phases, but $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 =$ baryon # has no impact $\Rightarrow$ remove 3 phases $\Rightarrow$ no $\cancel{CP}$ and one angle.

This is the "Cabibbo Angle"

$$\Rightarrow \quad V_{CKM}^{(2)} = \begin{pmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{pmatrix}$$

w/ $\sin\theta_c \simeq 0.23$

- Assume $N=3$ generations $\Rightarrow$ 3 angles and 6 phases. Can use 5 of 6 phases $\vec{\alpha} + \vec{\beta}$ to remove 3 phases. $\Rightarrow$ V_{CKM} specified by 3 angles and 1 CP phase.

Now that we know the physical content of the CKM matrix, we can study how it enters the Feynman rules. There are two basis choices that are typically discussed. The first is the "gauge eigenstate" basis, where the couplings to the W boson are diagonal in flavor space, the up-type quark masses are diagonal, but the down-type Yukawas take the form (in unitary gauge):

$$\mathcal{L} \supset -\,\vec{\bar{Q}}_L H \cdot V_{CKM} \cdot Y_d^D\, \vec{d}_R + h.c.$$

$$= -\frac{v+h}{\sqrt{2}}\, \vec{\bar{d}}_L \cdot V_{CKM} \cdot Y_d^D\, \vec{d}_R + h.c.$$

$\Rightarrow$ non-diagonal mass matrix, so mixing effects must be included in propagators.

The other basis is the "mass eigenstate" basis, where we diagonalize the down-type Yukawa couplings:

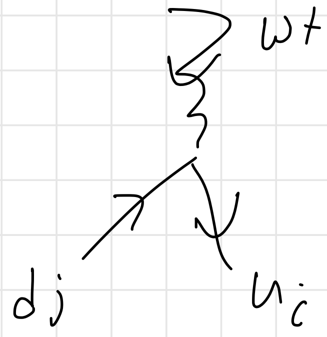
$$\begin{pmatrix} d_{L,1}^{(g)} \\ d_{L,2}^{(g)} \\ d_{L,3}^{(g)} \end{pmatrix} = V_{CKM} \begin{pmatrix} d_L \\ s_L \\ b_L \end{pmatrix}$$

$\leftarrow$ gauge basis

$$\Rightarrow \mathcal{L} \supset \frac{g}{\sqrt{2}} (V_{CKM})_i{}^j W_\mu^+ \bar{u}_L{}^i \gamma^\mu d_{L,j}$$

$$+ \frac{g}{\sqrt{2}} (V_{CKM}^*)_j{}^i W_\mu^- \bar{d}_L{}^j \gamma^\mu u_{L,i}$$

All other interactions are flavor diagonal.

$\Rightarrow$ 

$$= i\frac{g}{\sqrt{2}} (V_{CKM})_{ij}\, \gamma^\mu (1-\gamma^5)$$

+ conjugate diagram

The CKM matrix is often written as

$$V_{CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

So that 

$\sim V_{us}$

There is a common phenomenological parametrization due to Wolfenstein:

$$V_{CKM} = \begin{pmatrix} 1 - \lambda^2/2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1 - \frac{\lambda^2}{2} & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} + O(\lambda^2)$$

w/ $\lambda \simeq \sin\theta_c \simeq 0.23$

$A \simeq 0.8$, $\rho \simeq 0.14$, $\eta \simeq 0.35$

$\Rightarrow$ $1 \leftrightarrow 2$ transitions most likely, $2 \leftrightarrow 3$ somewhat likely, $1 \leftrightarrow 3$ transitions least likely

Since $\cancel{CP}$ requires 3-families, we can put the phase in 1-3 entry ⇒ tiny effect. 4.49

It is often a good approx to take

$$V_{CKM} \simeq \begin{pmatrix} V_{Cabibbo} & 0 \\ 0 & 1 \end{pmatrix}$$